

INTEGRABILITY AND REDUCTION OF POISSON GROUP ACTIONS ¹

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Abstract. In this paper we study Poisson actions of complete Poisson groups, without any connectivity assumption or requiring the existence of a momentum map. For any complete Poisson group G with dual G^* we obtain a suitably connected integrating symplectic double groupoid \mathcal{S} . As a consequence, the cotangent lift of a Poisson action on an integrable Poisson manifold P can be integrated to a Poisson action of the symplectic groupoid $\mathcal{S} \rightrightarrows G^*$ on the symplectic groupoid for P . Finally, we show that the quotient Poisson manifold P/G is also integrable, giving an explicit construction of a symplectic groupoid for it, by a reduction procedure on an associated morphism of double Lie groupoids.

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Introduction

Symplectic groupoids were introduced independently by Karasëv [16], Weinstein [23] and Zakrzewski [27] in the attempt of providing a geometric setting for the quantization of Poisson manifolds. The symplectic foliation of an integrable Poisson bivector is described by the orbits of an integrating symplectic groupoid, therefore symplectic groupoids also represent a fundamental tool in the study of the classical geometry of Poisson manifolds.

In recent years, the integration problem for Poisson manifolds to symplectic groupoids has been completely solved. Mackenzie and Xu showed in [14] that a Poisson manifold P is integrable to a symplectic groupoid if and only if the associated Lie algebroid on T^*P is integrable. Cattaneo and Felder produced in [2] a topological model for the symplectic groupoid of *any* Poisson manifold, yielding a smooth symplectic groupoid in the integrable case, by the symplectic reduction of the Poisson sigma model. Finally Crainic and Fernandes obtained in [4, 5] necessary and sufficient conditions for a Poisson manifold to be integrable, i.e. to endow the topological groupoid of Cattaneo and Felder with a smooth structure.

On the one hand Poisson manifolds behave well under reduction for actions as general as those of Poisson groupoids: a Poisson bivector always descends to the quotient by a free and proper compatible action [24]. On the other hand, Poisson manifolds are not always integrable to symplectic groupoids; the natural question is thus

Are quotients of integrable Poisson manifolds also integrable?

A positive answer was recently given by Fernandes-Ortega-Ratiu [8] in the case of Lie group actions and Lu [21] gave a construction of symplectic groupoids for certain Poisson homogeneous spaces. In the case of Poisson group actions with a complete momentum mapping, Xu described in [25] a reduction procedure on a lifted moment map; when the reduced space of the latter is smooth, it is a symplectic groupoid for the quotient Poisson structure.

In this paper we give a positive answer to the above question in the case of free and proper Poisson actions of complete Poisson groups. Under these assumptions on a Poisson G -space P , with symplectic groupoid Λ , the integrability of the left dressing action $\mathfrak{g}^* \rightarrow \mathfrak{X}(G)$ allows to lift the action of G on P to the action of suitably connected symplectic double groupoid on a canonical morphism of Poisson groupoids $\mathcal{J} : \Lambda \rightarrow G^*$ to the dual Poisson group. Moreover \mathcal{J} extends to a “moment morphism” of double Lie groupoids, whose kernel (double Lie groupoid) provides the data for a reduction *à la* Mikami-Weinstein [15] to produce an integration of the quotient Poisson bivector on P/G .

Our main result may be rephrased as the following

Theorem (5.2). *Let G be a complete Poisson group and P a Poisson G -space. If G acts freely and properly and P is integrable, the quotient Poisson manifold P/G is also integrable; in particular, the quotient $\mathcal{J}^{-1}(e_*)/G$ is smooth and the unique symplectic form $\underline{\omega}$ on $\mathcal{J}^{-1}(e_*)/G$ such that $\text{pr}^*\underline{\omega} = \iota^*\omega$ makes it a symplectic groupoid for their quotient Poisson bivector on P/G .*

On the way to our main result we also show that any complete Poisson group (not necessarily 0- or 1-connected) is integrable to a symplectic double groupoid (theorem 3.2), extending a result by Lu and Weinstein [18]. Moreover, we also produce a class of examples for which our functorial approach to the integrability of Lie algebroids [22] is effective; namely, all action \mathcal{LA} -groupoids associated with Poisson actions of complete Poisson groups on integrable Poisson manifolds are integrable by suitable action double Lie groupoids (theorem 4.1).

Mackenzie’s approach to Poisson reduction [12] has been a major source of inspiration for this paper; in fact, we develop here an integrated version of the reduction procedure described in [12].

The reduction of morphisms of Poisson groupoids by compatible actions of symplectic double groupoids can be performed along the lines of this paper. This approach is however ineffective to obtain symplectic groupoids for quotients of Poisson groupoid actions. We shall discuss in a forthcoming paper the integrability of this type of quotient Poisson manifolds.

This paper is organized as follows. The first two sections are introductory. In §1 we present the main definitions and review the notion of duality for Poisson groupoids in relation with integrability issues². In §2 we fix our conventions for Lie algebroid homotopies and recall the lifting conditions of [22] for the integrability of fibred products of Lie algebroids. In §3 we discuss the integrability of complete Poisson groups to symplectic double groupoids. In §4 we lift the action of a complete Poisson group to a compatible action of the associated symplectic double groupoid. In §5 we perform the reduction of the “moment morphism”. We finally comment on Xu’s reduction of [25] and Fernandes-Ortega-Ratiu’s integration of [8], which we obtain as special cases. In particular our construction shows that Xu’s quotient is always smooth; we conclude by remarking that the condition of [8] for integration to commute with reduction extends to our setting.

We fix at the end of this paper some of our conventions and special notations.

²In this paper, apart from §1, we shall only consider dual Poisson groups. We chose to define duality in the context of Poisson groupoids, since it allows us to introduce quickly and more naturally \mathcal{LA} -groupoids and double Lie groupoids, which we use throughout the rest of the paper.

1. Duality for Poisson groupoids

A Poisson groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a Poisson bivector Π making the graph $\Gamma(\mu)$ of the partial multiplication $\mu : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ coisotropic in $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$. When Π is non degenerate, hence $\Pi^\sharp = \omega^{\sharp-1}$ for some symplectic form ω , the compatibility condition is equivalent to $\Gamma(\mu)$ being Lagrangian and \mathcal{G} is called a **symplectic groupoid**. A Poisson groupoid over the one point manifold is a **Poisson group** and its multiplication is a Poisson map. Any Lie group(oid) is a Poisson group(oid) for the zero Poisson bivector.

The main properties of Poisson groupoids were proved in [24]; in particular the unit section $M \rightarrow \mathcal{G}$ is a coisotropic embedding (a Lagrangian embedding in the symplectic case) and there exists a unique Poisson structure on M making the source map Poisson. We shall say that the special Poisson structure on the base of a Poisson groupoid is the **induced Poisson structure**. Consider now any Poisson manifold (P, π) ; there exists a unique Lie algebroid bracket on $\Omega^1(P)$ extending the Lie bracket

$$[df_+, df_-] := d\{f_+, f_-\} \quad , \quad f_{\pm} \in \mathcal{C}^\infty(P) \quad ,$$

on the space of exact 1-forms according to the Leibniz rule, that is compatible with the Poisson anchor $\pi^\sharp : T^*P \rightarrow TP$. Whenever such a Lie algebroid is integrable to a Lie groupoid $\Lambda \rightrightarrows P$, one can show [14] that Λ , provided it is source 1-connected, carries a unique symplectic form making it a symplectic groupoid and inducing π on the base manifold P . In that case P is called an **integrable Poisson manifold** and Λ is called *the* symplectic groupoid of P .

Let us describe a typical example of an integrable Poisson manifold. For any Lie group G with Lie algebra \mathfrak{g} , T^*G can be endowed with a Lie groupoid structure over \mathfrak{g}^* making it a symplectic groupoid for the linear Poisson bivector on \mathfrak{g}^* dual to the Lie bracket of \mathfrak{g} (the construction applies, mutatis mutandis, to any Lie groupoid \mathcal{G} with Lie algebroid A to produce a symplectic groupoid $T^*\mathcal{G} \rightrightarrows A^*$, see [3] for details). Source \hat{s} and target \hat{t} are the evaluations of the left and right vector bundle trivializations $T^*G \rightarrow \mathfrak{g}^* \times G$, the inversion is the anti-transpose $\hat{i} = -d\iota^t$ of the tangent group inversion and the unit section \hat{e} is the identification of \mathfrak{g}^* with the conormal bundle N^*e of the identity element $e \in G$. One can check that the conormal bundle $N^*\Gamma(\mu) \subset T^*G^{\times 3}$ of the group multiplication is the graph of a map, therefore setting $\Gamma(\hat{\mu}) := (\text{id}_{T^*G} \times \text{id}_{T^*G} \times -\text{id}_{T^*G})N^*\Gamma(\mu)$, yields a groupoid multiplication $\hat{\mu} : (T^*G)^{(2)} \rightarrow T^*G$, whose graph is Lagrangian in $T^*G \times T^*G \times \overline{T^*G}$, by construction, therefore making $T^*G \rightrightarrows \mathfrak{g}^*$ symplectic groupoid. The Poisson bracket on \mathfrak{g}^* is induced by the anti-canonical symplectic form.

Let now (G, Π) be a Poisson group. The linearization of Π at the identity endows \mathfrak{g}^* with a compatible Lie bracket $[\cdot, \cdot]$,

$$\langle [\xi_+, \xi_-], x \rangle := (\mathcal{L}_{\overleftarrow{x}} \Pi)_e(\xi_+, \xi_-) \quad , \quad \xi_{\pm} \in \mathfrak{g}^* \quad , \quad x \in \mathfrak{g} \quad ,$$

for the left invariant vector field \overleftarrow{x} of $x \in \mathfrak{g}$. Such a pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a Lie bialgebra. We shall remark that the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ has a natural Lie algebra structure, the Drinfel'd double \mathfrak{d} , obtained as a double twist of the canonical bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$, making \mathfrak{g} and \mathfrak{g}^* Lie subalgebras and the canonical Riemannian structure ad-invariant³.

More generally Poisson groupoids differentiate to Lie bialgebroids: a Lie bialgebroid (A, A^*) [13] is a pair of Lie algebroids in duality such that the Lie algebroid differential $\Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+1} A)$ induced by A^* is a derivation of the graded Lie bracket on $\Gamma(\wedge^\bullet A)$ induced by A [17]. The source 1-connected integration of either Lie algebroid of a Lie bialgebroid, when it exists, can be endowed with a unique Poisson structure making it a Poisson groupoid and inducing the given Lie bialgebroid [14].

$$\begin{array}{ccc} \mathcal{D} & \rightrightarrows & \mathcal{V} \\ \Downarrow & & \Downarrow \\ \mathcal{H} & \rightrightarrows & M \end{array}$$

Figure 1

$$\begin{array}{ccc} \Omega & \rightrightarrows & A \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$

Figure 2

$$\begin{array}{ccc} T^*G & \rightrightarrows & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ G & \rightrightarrows & \bullet \end{array}$$

Figure 3

$$\begin{array}{ccc} T^*\mathcal{G} & \rightrightarrows & A^* \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$

Figure 4

On the other hand Poisson groupoids, which have integrable Poisson structure, integrate to symplectic double groupoids, at least in some favorable cases. A double Lie groupoid [9] (fig. 1) is a groupoid object in the category of groupoids, such that all sides are Lie groupoids and the horizontal structural maps are morphisms of Lie groupoids over the side horizontal maps for the vertical Lie groupoid structures. In order to make the domains of the top multiplications Lie groupoids over the domains of the side multiplications, the double source map $\mathbb{S} : \mathcal{D} \rightarrow \mathcal{H}_{s_h} \times_{s_v} \mathcal{V}, d \mapsto (s_v(d), s_h(d))$ is required to be submersion⁴. A symplectic double groupoid is a double Lie groupoid endowed with a symplectic form making both the top groupoids $\mathcal{D} \rightrightarrows \mathcal{V}$ and $\mathcal{D} \rightrightarrows \mathcal{H}$ symplectic groupoids. It follows [10] that the induced Poisson structures on the side groupoids of a symplectic double groupoid \mathcal{D} as above are Poisson groupoids in duality, in the sense that the corresponding Lie bialgebroids are isomorphic up to a canonical flip operation: if (A, A^*) is the Lie bialgebroid of \mathcal{H} , the Lie bialgebroid of \mathcal{V} is canonically isomorphic to $(A^*, -A)$, where $-A$ is the Lie algebroid obtained by changing the signs of bracket and anchor of A . For any Poisson groupoid

³For an account of the basic facts on Poisson groups and Lie bialgebras see [6].

⁴In [9] the double source map is also required to be surjective. We drop the condition, since it plays a crucial role only in the theory of transitive double Lie groupoids.

$(\mathcal{G}, \Pi) \rightrightarrows M$ with integrable Poisson structure it is easy to see that A^* is an integrable Lie algebroid, thus the source 1-connected integration $\mathcal{G}^* \rightrightarrows M$ carries a unique Poisson structure Π^* making it a Poisson groupoid with Lie bialgebroid $(A^*, -A)$. Following the conventions of [10, 22], we shall say that $(\mathcal{G}^*, \Pi^*) \rightrightarrows M$ is the (weak) dual of the given Poisson groupoid⁵. A double of a Poisson groupoid is a symplectic double groupoid with the given Poisson groupoid and the unique dual as side Poisson groupoids: such a double provides a stronger duality relation between its side groupoids.

A double Lie groupoid is a groupoid object in the category of Lie groupoids and applying the Lie functor to the its vertical groupoids yields an \mathcal{LA} -groupoid [9], i.e a diagram such as that in figure 2, where both horizontal sides are Lie groupoids and both vertical sides are Lie algebroids, making the top groupoid structural maps morphisms of Lie algebroids over the side ones; analogously to the case of double Lie groupoids, the double source map $\$: \Omega \rightarrow \mathcal{G}_s \times_{\text{pr}} A$, $\omega \mapsto (\text{Pr}(\omega), \hat{s}(\omega))$ is required to be surjective, equivalently the top source map to be fibrewise surjective, to make the domain of the top multiplication a Lie algebroid over the domain of the side multiplication. If G is a Poisson group, the diagram in figure 3 is a typical example of an \mathcal{LA} -groupoid. Note that the source regularity condition is fulfilled since the top source map is fibrewise an isomorphism of vector spaces.

Following a functorial approach, one can see [22] that \mathcal{LA} -groupoids with integrable top Lie algebroid do integrate to double Lie groupoids, provided some lifting conditions on the top source map of the \mathcal{LA} -groupoid are met. The result specializes to the \mathcal{LA} -groupoid (fig. 4) of a Poisson groupoid and one can further show that the integrating double Lie groupoid is indeed a double of the original Poisson groupoid (see [10] for the definition of the \mathcal{LA} -groupoid structure). We shall check in §3 our lifting conditions in the case of a complete Poisson group.

2. Lifting of Lie algebroid homotopies

Recall from [4], that, for any Lie algebroid $A \rightarrow M$ with anchor ρ , an A -path is a \mathcal{C}^1 map $\alpha : I \rightarrow A$, over a \mathcal{C}^2 base path $\gamma : I \rightarrow M$, such that $d\gamma = (\rho \circ \gamma)\alpha$, that is, a morphism of Lie algebroids $TI \rightarrow A$. For any Lie groupoid \mathcal{G} , a \mathcal{G} -path is a \mathcal{C}^2 path within a source fibre, starting from the unit section. Whenever \mathcal{G} is source connected with Lie algebroid A , the right derivative $\delta_r : \{\mathcal{G}\text{-paths}\} \rightarrow \{A\text{-paths}\}$, $\delta_r g(u) = \text{dr}_{g(u)}^{-1} \dot{g}(u)$, yields an homeomorphism for the natural Banach topologies. Homotopy of \mathcal{G} -paths within the source fibres and relative to the endpoints, called for short \mathcal{G} -homotopy, can be translated to an equivalence relation for A -paths.

⁵Declaring the dual groupoid of an integrable Poisson groupoid to be that with Lie bialgebroid (A, A^*) is also frequent in the literature.

An A -homotopy from an A -path α_- to an A -path α_+ is a \mathcal{C}^1 morphism of Lie algebroids $h : TI^{\times 2} \rightarrow A$, satisfying suitable boundary conditions⁶. Regarding h as a 1-form taking values in the pullback X^+A , for the base map X , the anchor compatibility condition is $dX = (\rho \circ X)h$, while the bracket compatibility takes the form of a Maurer-Cartan equation

$$Dh + \frac{1}{2}[h \wedge h] = 0 \quad , \quad \iota_{\partial_{\mathbb{H}}^{\pm}}^* h = \alpha_{\pm} \quad \text{and} \quad \iota_{\partial_{\mathbb{V}}^{\pm}}^* h = 0$$

where D is the covariant derivative of the pullback of an arbitrary connection ∇ for A and $[\theta_+ \wedge \theta_-]$ is the contraction $-\iota_{\theta_+ \wedge \theta_-} X^+ \tau^{\nabla}$ with the pullback of the torsion tensor, $\theta_{\pm} \in \Omega^1(X^+A)$. It turns out that the quotients $\{A\text{-paths}\}/A\text{-homotopy}$ and $\{\mathcal{G}\text{-paths}\}/\mathcal{G}\text{-homotopy}$ are both endowed with natural Lie groupoid structures, isomorphic to the source 1-connected cover of \mathcal{G} , the groupoid multiplications essentially being given by concatenation in both cases.

A morphism $\phi : A \rightarrow B$ of Lie algebroids has the [22]

l_0) $0\text{-}\mathcal{LA}$ -homotopy lifting property if, for any A -path α_- and B -path β_+ , which is B -homotopic to $\beta_- := \phi \circ \alpha_-$, there exists an A -path α_+ , which is A -homotopic to α_- and satisfies $\phi \circ \alpha_+ = \beta_+$;

l_1) $1\text{-}\mathcal{LA}$ -homotopy lifting property if, for any A -path α , which is A -homotopic to the constant A -path $\alpha_o \equiv 0_{\text{pr}_A(\alpha(0))}$, and B -homotopy h_B from $\beta := \phi \circ \alpha$ to the constant B -path $\beta_o \equiv 0_{\text{pr}_B(\beta(0))}$, there exists an A -homotopy h_A from α to α_o , such that $\phi \circ h_A = h_B$.

The \mathcal{LA} -homotopy lifting conditions above on a morphism of integrable Lie algebroids $A \rightarrow B$, translate to the infinitesimal level homotopy lifting conditions in the source fibres of the source 1-connected Lie groupoids \mathcal{A} and \mathcal{B} of A and B along the integration $\mathcal{A} \rightarrow \mathcal{B}$.

Proposition 2.1. *Let $A^{1,2} \rightarrow M^{1,2}$, $B \rightarrow N$ be integrable Lie algebroids and $\phi_{1,2} : A^{1,2} \rightarrow B$ be transversal morphisms of Lie algebroids over $f_{1,2} : M^{1,2} \rightarrow N$, so that the fibred product Lie algebroid $A^1_{\phi_1} \times_{\phi_2} A^2$ is defined; denote with $\varphi_{1,2} : \mathcal{G}^{1,2} \rightarrow \mathcal{H}$ the integrating morphisms for the source 1-connected integrations. Then the fibred product Lie groupoid $\mathcal{G}^1_{\varphi_1} \times_{\varphi_2} \mathcal{G}^2 \rightrightarrows M_1 \times_{f_1} \times_{f_2} M_2$ is source 1-connected provided either ϕ_1 or ϕ_2 has the 0- and 1- \mathcal{LA} -homotopy lifting property.*

This fact was proved in a (not so) special case in [22] (proposition 3.3.); the same proof applies here, with the obvious modifications. The transversality conditions are met, for example, when either ϕ_1 or ϕ_2 is fibrewise submersive and base submersive.

⁶The boundary components of the compact square $I^{\times 2}$ shall be denoted with $\partial_{\mathbb{V}}^{\pm} = \{(\varepsilon, 1/2 \pm 1/2)\}_{\varepsilon \in I}$, respectively with $\partial_{\mathbb{H}}^{\pm} = \{(1/2 \pm 1/2, u)\}_{u \in I}$; $\iota_{\partial} : I \rightarrow I^{\times 2}$ is the inclusion of the boundary component ∂ .

3. Dressing actions and symplectic double groupoids

Let (G, Π) be a Poisson group. The (left) dressing action of \mathfrak{g}^* on G is the infinitesimal action $\Upsilon : \mathfrak{g}^* \rightarrow \mathfrak{X}(G)$, $\Upsilon(\xi) := \Pi^\# \overleftarrow{\xi}$, where $\overleftarrow{\xi}$ is the left invariant 1-form on G associated with $\xi \in \mathfrak{g}^*$. The (left) dressing vector fields on G are those in the image of \mathfrak{g}^* under the left dressing action map. Left dressing vector fields do not have, in general, complete flows; when they have, e.g. when G is compact, G is called a **complete** Poisson group.

Remark 3.1. To any infinitesimal action $\sigma : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ one can associate a Lie algebroid $\mathfrak{g} \ltimes M$ over M , whose leaves are precisely the orbits of the infinitesimal action. The anchor $\rho_\ltimes : \mathfrak{g} \ltimes M \rightarrow TM$ is given by $\rho_\ltimes(x, q) := \sigma(x)_q$. Sections of $\mathfrak{g} \ltimes M$ are to be identified with \mathfrak{g} -valued functions over M , thus post-composition with the action map σ induces a map $\mathcal{C}^\infty(M, \mathfrak{g}) \rightarrow \text{End}_{\text{lin}} \mathcal{C}^\infty(M, \mathfrak{g})$, $f \mapsto \sigma^f$,

$$\begin{aligned} \langle \sigma_q^{f+}(f_-), \xi \rangle &:= (\sigma(f_+(q))_q(\langle \xi, f_- \rangle)) \\ &= \rho_\ltimes(f_-)_q(\langle \xi, f_+ \rangle) \quad , \quad \xi \in \mathfrak{g}^* \quad , \end{aligned}$$

$f_\pm \in \mathcal{C}^\infty(M, \mathfrak{g})$. The skewsymmetric bilinear operation

$$[f_+ \ltimes f_-]_q := [f_+(q)^*, f_-(q)] + \sigma_q^{f+}(f_-) - \sigma_q^{f-}(f_+) \quad , \quad q \in M \quad ,$$

defines a Lie algebroid bracket on $\mathfrak{g} \ltimes M$ compatible with ρ_\ltimes . Similarly, to any Lie group action $G \rightarrow \text{Diff}(M)$, one can associate an action Lie groupoid $G \ltimes M \rightrightarrows M$, whose orbits on M are the orbits of the action (we shall define action groupoids for a Lie group(oid) actions in §4).

In the case of the dressing action associated with a Poisson group G , the left trivialization $T^*G \rightarrow \mathfrak{g}^* \times G$ is an isomorphism of Lie algebroids to the action Lie algebroid. When G is complete, the infinitesimal dressing action integrates to a global action, $\tilde{\Upsilon} : G^* \times G \rightarrow G$ of the dual 1-connected Poisson group G^* on G and the action groupoid $G^* \ltimes G \rightrightarrows G$ is the source 1-connected integration of $\mathfrak{g} \ltimes G$. Provided G is 1-connected (in this case G^* is also complete) the same argument applies to the integration of the infinitesimal right dressing action $\mathfrak{g}^* \rightarrow \mathfrak{X}(G)$, defined using Π^* , to a global action $G^* \times G \rightarrow G^*$. One can show that the 1-connected integration D of the Drinfel'd double \mathfrak{d} is isomorphic to bitwisted product $G^* \ltimes G$ carrying two further symplectic groupoid structures over G and G^* [16, 19] and making $(G^* \ltimes G, G; G^*, \bullet)$ a double of the Poisson group on G . A construction of a double in the non-complete case, for a 1-connected G was given in [18] by Lu and Weinstein. We remark that the total space of Lu-Weinstein's double is only locally diffeomorphic to D , unless G is complete, and it is in general neither source (1-)connected over G , nor over G^* . Moreover, the 1-connectivity of G is essential in both constructions.

The rest of this section is devoted to improve the above results to a suitably simply connected integration, in the complete case, dropping the connectivity assumptions on G .

Theorem 3.2. *For any complete Poisson group G , the source 1-connected symplectic groupoid \mathcal{S} of G carries a unique Lie groupoid structure over the 1-connected dual Poisson group G^* making it a symplectic groupoid for the dual Poisson structure and for which*

$$(3.1) \quad \begin{array}{ccc} \mathcal{S} & \rightrightarrows & G^* \\ \Downarrow & & \Downarrow \\ G & \rightrightarrows & \bullet \end{array}$$

is a symplectic double groupoid.

The symplectic double groupoid of last theorem is then a vertically source 1-connected double of G . We derive this result from the functorial approach to the integrability of \mathcal{LA} -groupoids developed in [22]; we shall recall below our approach in the special case of the \mathcal{LA} -groupoid (fig. 3) of a Poisson group.

Let $\mathcal{S} \rightrightarrows G$ be the source 1-connected symplectic groupoid of G . The groupoid structural maps \hat{s} , \hat{t} , $\hat{\varepsilon}$ and $\hat{\iota}$ of the cotangent prolongation groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ integrate uniquely to morphisms of Lie groupoids $s_H, t_H : \mathcal{S} \rightarrow G^*$, $\varepsilon_H : G^* \rightarrow \mathcal{S}$ and $\iota_H : \mathcal{S} \rightarrow \mathcal{S}$. The nerves

$$(T^*G)^{(n+1)} = T^*G_{\hat{s}} \times_{\hat{t} \circ \text{pr}_1} (T^*G)^{(n)} \quad , \quad n \in \mathbb{N} \quad ,$$

of the top groupoid in figure 2, are canonically endowed with fibred product Lie algebroids over the side nerves $G^{(n)} = G^{\times n}$ and integrate to the nerves of the top horizontal differentiable graph $(\mathcal{S}, G^*; s_H, t_H)$

$$\mathcal{S}_H^{(n+1)} = \mathcal{S}_{s_H} \times_{t_H \circ \text{pr}_1} \mathcal{S}_H^{(n)} \quad , \quad n \in \mathbb{N} \quad .$$

Provided the nerves $\mathcal{S}_H^{(\bullet)}$ are source 1-connected, the top groupoid multiplication $\hat{\mu} : (T^*G)^{(2)} \rightarrow T^*G$, integrates uniquely to a morphism of Lie groupoids $\mu_H : \mathcal{S}_H^{(2)} \rightarrow \mathcal{S}$. One can check that it is indeed a groupoid multiplication compatible with $(s_H, t_H, \varepsilon_H, \iota_H)$ by integrating the diagrams for the groupoid structure on $T^*G \rightrightarrows \mathfrak{g}^*$. In particular, it is possible to integrate the associativity diagram, since the third nerve $\mathcal{S}_H^{(3)}$ is source 1-connected. Moreover, the graph $\Gamma(\hat{\mu})$ carries a Lie algebroid structure over $\Gamma(\mu)$ making it a Lagrangian subalgebroid of $T^*G \times T^*G \times \overline{T^*G}$; $\Gamma(\mu_H) \simeq \mathcal{S}_H^{(2)}$ is the corresponding source 1-connected integration, therefore [1] a Lagrangian subgroupoid of $\mathcal{S} \times \mathcal{S} \times \overline{\mathcal{S}}$. Thus (3.1) is indeed a symplectic double groupoid and the induced Poisson structure on G^* coincides with Π^* .

To complete the proof of theorem 3.2, we need to show that the nerves $\mathcal{S}_H^{(\bullet)}$ are actually source 1-connected. This is consequence of proposition 2.1 and the following

Lemma 3.3. *For any complete Poisson group G , the cotangent source map $\hat{s} : T^*G \rightarrow \mathfrak{g}^*$ has the 0- and 1- \mathcal{LA} -homotopy lifting properties.*

Proof. Note that the diagram

$$\begin{array}{ccc} T^*G & \xrightarrow{\sim} & \mathfrak{g}^* \ltimes G \\ & \searrow \hat{s} & \swarrow \text{pr} \\ & \mathfrak{g}^* & \end{array}$$

commutes in the category of Lie algebroids for the left trivialization on the top edge: it suffices to prove the statement for the projection $\mathfrak{g}^* \ltimes G \rightarrow \mathfrak{g}^*$. $\mathfrak{g}^* \ltimes G$ -paths are pairs (ξ, γ) of \mathfrak{g}^* -paths and paths in G , such that $\Upsilon(\xi(u))_{\gamma(u)} = d\gamma_u$, $u \in I$. A Lie algebroid homotopy $h_{\ltimes} : TI^{\times 2} \rightarrow \mathfrak{g}^* \ltimes G$ is a pair (h, X) for which h is a \mathfrak{g}^* -homotopy and $X : I \rightarrow G$ satisfies $dX = \Upsilon \circ h$. To see this, note that h_{\ltimes} takes values in the pullback bundle $X^+(\mathfrak{g}^* \ltimes G) \simeq \mathfrak{g}^* \times I^{\times 2}$. The bracket compatibility for h_{\ltimes} can be written choosing the de Rham differential on G with coefficients in \mathfrak{g}^* as a linear connection ∇ for $\mathfrak{g}^* \ltimes G$; this way the covariant derivative for the pullback connection is simply the de Rham differential on $I^{\times 2}$ with coefficients in \mathfrak{g}^* . The torsion tensor of ∇ is

$$\begin{aligned} \tau^\nabla(f_+, f_-) &= df_{-g}(\rho^\ltimes(f_+)) - df_{+g}(\rho^\ltimes(f_-)) - [f_+ \ltimes f_-]_g \\ &= -[f_+(g)^*, f_-(g)] \quad , \end{aligned}$$

$f_\pm \in \mathcal{C}^\infty(G, \mathfrak{g}^*)$, thus the pullback of τ^∇ induces the canonical graded Lie bracket, also denoted by $[\cdot, \cdot]$, on $\Omega^\bullet(I^{\times 2}, \mathfrak{g}^*)$ and the bracket compatibility condition for h_{\ltimes} reduces to the classical Maurer-Cartan equation

$$dh + \frac{1}{2}[h^*, h] = 0$$

for $h \in \Omega^1(I^{\times 2}, \mathfrak{g}^*)$. Suppose now (ξ_-, γ_-) is a fixed $\mathfrak{g}^* \ltimes G$ -path, let h be a \mathfrak{g}^* -homotopy from ξ_- to some other \mathfrak{g}^* -path ξ_+ and $H : I^{\times 2} \rightarrow G^*$ the unique G^* -homotopy integrating h , i.e. such that

$$h = \delta_r^\varepsilon H \cdot d\varepsilon + \delta_r^u H \cdot du$$

for the partial right derivatives. We claim that $h_{\ltimes} := (h, X)$, where

$$\begin{aligned} X(\varepsilon, u) &= H(\varepsilon, u) \cdot H(0, u)^{-1} * \gamma_-(u) \\ &= H(\varepsilon, u) * H(0, u)^{-1} * \gamma_-(u) \quad , \quad u, \varepsilon \in I \quad , \end{aligned}$$

is a $\mathfrak{g}^* \ltimes G$ -homotopy; here the symbol $*$ denotes the integrated dressing action. We postpone to the appendix this straightforward but lengthy check.

The lifting conditions follow: by construction

$$\begin{aligned}\iota_{\partial_V^\pm}^*(h, X) &= (\iota_{\partial_V^\pm}^* h, X \circ \iota_{\partial_V^\pm}) = 0 \\ \iota_{\partial_H^\pm}^*(h, X) &= (\xi_\pm, \gamma_\pm)\end{aligned}$$

for some path γ_+ in G , (l_0) If h is a fixed \mathfrak{g}^* -homotopy to some fixed \mathfrak{g}^* -path ξ_+ , (ξ_+, γ_+) is the desired lift. (l_1) In particular if ξ_- is $\mathfrak{g}^* \times G$ -homotopic to the constant $\mathfrak{g}^* \times G$ -path $\xi_o \equiv 0$ and h a \mathfrak{g}^* -homotopy from ξ_- to the constant \mathfrak{g}^* -path $\xi_+ \equiv 0$, we have

$$d\gamma_+|_u = \Upsilon(\xi_+(u))_{\gamma_+(u)} = 0_{\gamma_+(u)} \quad ,$$

hence $\gamma_+(u) \equiv \gamma_+(0) = \gamma_-(0)$, since the base paths of homotopic Lie algebroid paths are homotopic relatively to the endpoints; (h, X) is then the desired homotopy. \square

4. Integrability of Poisson G -spaces

A Lie groupoid action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a smooth map $j : P \rightarrow M$ is given by a smooth map $\sigma : \mathcal{G}_s \times_j P \rightarrow P$, $(g, p) \mapsto g * p$, satisfying the conditions

$$(4.1) \quad j(g * p) = t(g) \quad \varepsilon(j(n)) * n = n \quad g * (h * p) = (g \cdot h) * p \quad ,$$

to be understood whenever they make sense, that generalize the notion of a Lie group action. The map j above is called **moment map** of the action. To any Lie groupoid action one can associate an action Lie groupoid $\mathcal{G} \ltimes P \rightrightarrows P$ with total space $\mathcal{G}_s \times_j P$, whose the source map is the first projection, the target map coincides with the action map, the partial multiplication is given by

$$(g_+, p_+) \cdot_\ltimes (g_-, p_-) = (g_+ \cdot g_-, p_-) \quad ,$$

for all elements of \mathcal{G} and P for which the expression above makes sense; unit section and inversion are defined accordingly.

When \mathcal{G} is a Poisson groupoid, P a Poisson manifold and j a Poisson map for the induced Poisson structure on M , P is called a **Poisson \mathcal{G} -space** if

$$(4.2) \quad \Gamma(\sigma) \subset \mathcal{G} \times P \times \overline{P}$$

is a coisotropic submanifold; in that case one calls the action a **Poisson action**. The same definition applies when \mathcal{G} is a Poisson group or a symplectic groupoid acting on a symplectic manifold. The compatibility condition is equivalent to require the action map $G \times P \rightarrow P$ to be Poisson, in first case, and to $\Gamma(\sigma)$ being Lagrangian, in the second, this makes the action a *symplectic groupoid action* in the sense of Mikami and Weinstein [15].

Let P be a Poisson G -space; dualizing the infinitesimal action $\sigma_o : \mathfrak{g} \rightarrow \mathfrak{X}(P)$, yields a morphism of Lie bialgebroids $T^*P \rightarrow \mathfrak{g}^*$,

$$(4.3) \quad \langle \hat{j}(\alpha_p), x \rangle := \langle \alpha_p, \sigma_o(x) \rangle \quad , \quad x \in \mathfrak{g} \quad ,$$

i.e. \hat{j} is at the same time a morphism of Lie algebroids and a Poisson map for the dual Poisson structures [26]. Remarkably, the G -action cotangent lifts to a Poisson action of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ on the moment map \hat{j} ; moreover the lifted action is compatible with the Lie algebroids on T^*G and T^*P [12]. In fact, the lifted action map $\hat{\sigma} : T^*G \times_{\hat{j}} T^*P \rightarrow T^*P$, $\theta_g * \alpha_p := \hat{\sigma}(\theta_g, \alpha_p) := \alpha_p \circ d\sigma_{g^{-1}}$, is a morphism of Lie algebroids and it follows that

$$(4.4) \quad \begin{array}{ccc} T^*G \times T^*P & \rightrightarrows & T^*P \\ \downarrow & & \downarrow \\ G \times P & \rightrightarrows & P \end{array}$$

is an \mathcal{LA} -groupoid, for the action groupoid of the cotangent lifted action on the top horizontal side and the fibred product Lie algebroid on the top vertical side.

Mackenzie's construction of (4.4) should be regarded as the lift of a Poisson group action to a morphic action in the category of Lie algebroids. By analogy, it is natural to consider morphic actions in the category of Lie groupoids, that is, pairs of groupoid actions $(\tilde{\sigma}, \sigma)$ on pairs of moment maps (\mathcal{J}, j) , such that both (\mathcal{J}, j) and $(\tilde{\sigma}, \sigma)$ form morphisms of Lie groupoids (fig. 5, 6). Thanks to the regularity condition on the double source map of \mathcal{D} , the fibred product $\mathcal{D}_{s_H} \times_{\mathcal{J}} \mathcal{B} \rightrightarrows \mathcal{H}_{s_h} \times_j N$ is always a Lie groupoid; moreover the diagram in figure 7 is a double Lie groupoid, we shall refer to as the double Lie groupoid of the action, for the action groupoids on the horizontal edges.

$$\begin{array}{ccc} \mathcal{D}_{s_H} \times_{\mathcal{J}} \mathcal{B} & \xrightarrow{\tilde{\sigma}} & \mathcal{B} \\ \Downarrow & & \Downarrow \\ \mathcal{H}_{s_h} \times_j N & \xrightarrow{\sigma} & N \end{array}$$

Figure 5

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathcal{J}} & \mathcal{V} \\ \Downarrow & & \Downarrow \\ N & \xrightarrow{j} & M \end{array}$$

Figure 6

$$\begin{array}{ccc} \mathcal{D} \times \mathcal{B} & \rightrightarrows & \mathcal{B} \\ \Downarrow & & \Downarrow \\ \mathcal{H} \times N & \rightrightarrows & N \end{array}$$

Figure 7

It is clear that a morphic action in the category of Lie groupoids differentiates to a morphic action in the category of Lie algebroids. A natural question is then whether the cotangent lift of a Poisson G -space integrates to a morphic action of a symplectic double groupoid integrating G on the integration $\mathcal{J} : \Lambda \rightarrow G^*$ of (4.3). The answer is positive in the complete case.

Theorem 4.1. *Let G be a complete Poisson group and P an integrable Poisson manifold with source 1-connected symplectic groupoid Λ . If P is a Poisson G -space, then*

- i) $\mathcal{J} : \Lambda \rightarrow G^*$ is a Poisson \mathcal{S} -space for the top horizontal groupoid $\mathcal{S} \rightrightarrows G^*$ of the vertically source 1-connected double of G ;*
- ii) The action map $\tilde{\sigma} : \mathcal{S}_{s_H} \times \mathcal{J} \Lambda \rightarrow \Lambda$ is a morphism of Lie groupoids over the action map $\sigma : G \times P \rightarrow P$;*
- iii) The action double groupoid of the integrated action*

$$(4.5) \quad \begin{array}{ccc} \mathcal{S} \ltimes \Lambda & \rightrightarrows & \Lambda \\ \Downarrow & & \Downarrow \\ G \ltimes P & \rightrightarrows & P \end{array}$$

is the vertically source 1-connected double Lie groupoid integrating the \mathcal{LA} -groupoid (4.4) associated with the Poisson action.

Proof. It was proved in [26] that \mathcal{J} is Poisson. We claim that the fibred product Lie groupoids $\mathcal{S}_{s_H} \times \mathcal{J} \Lambda$ and

$$\mathcal{S}_{s_H} \times \mathcal{J} (\mathcal{S}_{s_H} \times \mathcal{J} \Lambda) \simeq (\mathcal{S}_{s_H} \times {}_{t_H} \mathcal{S})_{s_H \circ \text{pr}_2} \times \mathcal{J} \Lambda$$

are source 1-connected; then $\hat{\sigma} \circ (\text{id}_{T^*G} \times \hat{\sigma})$, and $\hat{\sigma} \circ (\hat{\mu} \times \text{id}_{T^*P})$ integrate to the morphisms $\tilde{\sigma} \circ (\text{id}_{\mathcal{S}} \times \tilde{\sigma})$ and $\tilde{\sigma} \circ (\mu_H \times \text{id}_{\Lambda})$, which coincide up to the identification of the domains, by uniqueness. Compatibility of $\tilde{\sigma}$ with \mathcal{J} , follows in the same fashion. (ii) holds by construction and (iii) is now obvious. (i) $\Gamma(\tilde{\sigma}) \simeq \mathcal{S}_{s_H} \times \mathcal{J} \Lambda$ is the source 1-connected integration of

$$\Gamma(\hat{\sigma}) = (\text{id}_{T^*G} \times \text{id}_{T^*P} \times -\text{id}_{T^*P}) N^* \Gamma(\sigma) \quad ,$$

which is a Lagrangian Lie subalgebroid of $T^*G \times T^*P \times \overline{T^*P}$, by coisotropy of $\Gamma(\sigma)$ in $\mathcal{G} \times P \times \overline{P}$; it follows [1] that $\Gamma(\tilde{\sigma}) \subset \mathcal{S} \times \Lambda \times \overline{\Lambda}$ is a Lagrangian subgroupoid. Our claim follows from proposition 2.1 if the projection $T^*G \ltimes T^*P \rightarrow T^*P$ has the lifting properties (l_0, l_1) . Note that $T^*G \ltimes T^*P$ -paths, -homotopies, are pairs (X^G, X^P) of T^*G -paths, -homotopies, and T^*P -paths, -homotopies, such that $\hat{s} \circ X^G = \hat{j} \circ X^P$. For any assigned $(T^*G \ltimes T^*P)$ -path (ξ_-^G, ξ_-^P) and T^*P -homotopy h^P from ξ_-^P to some T^*P -path ξ_+^P , $\hat{j} \circ h^P$ is a \mathfrak{g}^* -homotopy from $\hat{s} \circ \xi_-^G$ to some \mathfrak{g}^* -path ξ_+ ; hence by applying lemma 3.3 $\hat{j} \circ h^P$ can be lifted to a T^*G -homotopy h^G such that $\hat{s} \circ h^G = \hat{j} \circ h^P$ and $\iota_{\partial_H^-}^* h^G = \xi_-^G$. Thus the pair $H^\times := (h^G, h^P)$ is a $T^*G \ltimes T^*P$ -homotopy lifting h^P , such that $\iota_{\partial_H^-}^* H^\times = (\xi_-^G, \xi_-^P)$; as in the proof of lemma 3.3, the boundary conditions necessary to fulfill (l_0, l_1) hold according to the initial data. \square

Remark 4.2. For any symplectic groupoid action such as above, the multiplicativity condition

$$(4.6) \quad \tilde{\sigma}^* \omega = \text{pr}_S^* \Omega + \text{pr}_\Lambda^* \omega$$

on $\mathcal{S} \ltimes \Lambda$ is equivalent to $\Gamma(\tilde{\sigma}) \subset \mathcal{S} \times \Lambda \times \overline{\Lambda}$ being Lagrangian, for the symplectic forms Ω on \mathcal{S} and ω on Λ . Explicitly, (4.6) reads

$$\omega_{s*\lambda}(\delta s_+ * \delta \lambda_+, \delta s_+ * \delta \lambda_+) = \Omega_s(\delta s_+, \delta s_+) + \omega_\lambda(\delta \lambda_+, \delta \lambda_+) \quad ,$$

where we have used the symbol $*$ for the tangent lift of $\tilde{\sigma}$, for all composable $\delta s_\pm \in T\mathcal{S}$ and $\delta \lambda_\pm \in T\Lambda$.

5. Integrability of quotient Poisson structures

Consider any Poisson action $\sigma : G \times P \rightarrow P$ of a (not necessarily complete) Poisson group G . By coisotropy of $\Gamma(\sigma)$ the G -invariant functions on P form a Poisson subalgebra $\mathcal{C}^\infty(P)^G \subset \mathcal{C}^\infty(P)$. When the quotient P/G is smooth, i.e. if the action is free and proper, its algebra of functions coincides with $\mathcal{C}^\infty(P)^G$ and can be endowed with a Poisson bracket simply by setting

$$\{f_+, f_-\}_{P/G}([p]_{P/G}) = \{f_+, f_-\}_P(p) \quad , \quad f_\pm \in \mathcal{C}^\infty(P)^G \quad ,$$

for any choice of a representative $p \in [p]_{P/G} \in P/G$. Therefore we have

Proposition 5.1. [24] *Let a Poisson group G act on a Poisson manifold P freely and properly. If the action is Poisson, there exists a unique Poisson structure on P/G making the quotient projection a Poisson submersion.*

Next, we shall obtain a symplectic groupoid for the quotient Poisson manifold, when P is integrable and G complete, using a reduction procedure on the action double groupoid of theorem 4.1; a few remarks are in order.

On the one hand, by the very definition (4.3) of the moment map \hat{j} , $\ker_p \hat{j} = N_p^* \mathcal{O}$, for the G -orbit \mathcal{O} through $p \in P$; thus, whenever the action is free,

$$\text{rank } \hat{j} = \dim P - (\dim P - \dim \mathcal{O}) = \dim \mathfrak{g}^*$$

and \hat{j} is a bundle map of maximal rank. It follows, if P is integrable, that the integrating morphism $\mathcal{J} : \Lambda \rightarrow G^*$ is sourcewise submersive and submersive, therefore the kernel groupoid $\ker \mathcal{J} = \mathcal{J}^{-1}(e_*)$ is a wide *Lie* subgroupoid.

$$\begin{array}{ccccc} \mathcal{S} \ltimes \Lambda & \rightrightarrows & \Lambda & & \\ \Downarrow & \text{pr}_\mathcal{S} & \Downarrow & \searrow \mathcal{J} & \\ G \ltimes P & \rightrightarrows & P & \xrightarrow{\quad} & S \rightrightarrows G^* \\ & & \searrow \hat{j} & & \Downarrow \\ & & G & \rightrightarrows & \bullet \end{array}$$

Figure 8

$$\begin{array}{ccc} G \ltimes \mathcal{J}^{-1}(e_*) & \rightrightarrows & \mathcal{J}^{-1}(e_*) \\ \Downarrow & & \Downarrow \\ G \ltimes P & \rightrightarrows & P \end{array}$$

Figure 9

On the other hand, the moment map for the integrated action can always be completed to a morphism of double Lie groupoids (fig. 8), whose (vertical) kernel (fig. 9) is naturally an action double groupoid for the restriction of the top action.

Theorem 5.2. *Let G be a complete Poisson group and P a Poisson G -space. If G acts freely and properly and P is integrable, the quotient Poisson manifold P/G is also integrable; in particular*

i) The quotient $\mathcal{J}^{-1}(e_)/G$ is smooth and the unique symplectic form $\underline{\omega}$ on $\mathcal{J}^{-1}(e_*)/G$ such that*

$$(5.1) \quad \text{pr}^* \underline{\omega} = \iota^* \omega$$

makes it a symplectic groupoid over P/G ;

ii) $\mathcal{J}^{-1}(e_)/G \rightrightarrows P/G$ integrates the quotient Poisson structure.*

Proof. Note that $\mathcal{K} := \mathcal{J}^{-1}(e_*)$ is coisotropic since \mathcal{J} is Poisson and $e_* \subset G^*$ is coisotropic. (i) There is a natural representation $\vartheta : G \rightarrow \text{Aut}(\mathcal{K})$, $\vartheta_g(\kappa) = \tilde{\sigma}(\varepsilon_V(g), \kappa) = g * \kappa$ in the group of Lie groupoid automorphisms, providing the descent data for the Lie groupoid structure. Since $P \rightarrow P/G$ is a principal G -bundle, it follows from ([11], lemma 2.1) that \mathcal{K}/G is a Lie groupoid over P/G . On $G \ltimes \mathcal{K}$ the multiplicativity condition reads

$$\omega_{g*\kappa}(\delta g_+ * \delta \kappa_+, \delta g_- * \delta \kappa_-) = \omega_\kappa(\delta \kappa_+, \delta \kappa_-) \quad ,$$

being $G \subset \mathcal{S}$ Lagrangian, for all composable $\delta g_\pm \in TG$ and $\delta \kappa_\pm \in T\mathcal{K}$; it follows that $\vartheta_g^*(\iota^* \omega) \equiv \iota^* \omega$, hence using (5.1) to define a closed 2-form on \mathcal{K}/G makes sense. We claim that the characteristic distribution Δ of \mathcal{K} spans $T_\kappa \mathcal{O}$ for the \mathcal{G} -orbit \mathcal{O} through $\kappa \in \mathcal{K}$; it follows that $\underline{\omega}$ is nondegenerate. To see this, note that

$$\begin{aligned} \text{rank } T^\omega \mathcal{K} &= \dim \Lambda - \text{rank } \mathcal{J} = \dim G^* \\ &= \dim \mathcal{O} \end{aligned}$$

and that all $\delta o \in T_k \mathcal{O}$ are of the form $\delta o = \delta g * 0_k$ for the tangent lifted action, thus, for all $\delta k \in T_k \mathcal{K}$,

$$\omega_k(\delta k, \delta g * 0_k) = \omega_k(\delta g^{-1} * \delta k, 0_k) = 0 \quad ,$$

i.e. for all $k \in \mathcal{K}$

$$\Delta_k = T_k^\omega \mathcal{K} = T_k \mathcal{O} \quad .$$

Multiplicativity of $\underline{\omega}$ follows easily from multiplicativity of ω and uniqueness is clear thanks to condition (5.1). (ii) Since the characteristic leaves of \mathcal{K} are the connected components of the G -orbits, $\mathcal{C}^\infty(\mathcal{K})^G \subset \mathcal{C}^\infty(\mathcal{K})^\Delta = \ker \Delta$, thus all extension $F_\pm \in \mathcal{C}^\infty(\Lambda)$ of $f_\pm \in \mathcal{C}^\infty(\mathcal{K})^G$ are in the normalizer of the vanishing ideal of \mathcal{K} and the Hamiltonian vector fields X^{F_\pm} restrict to \mathcal{K} .

Let $X^{F\pm} = d\iota Y^{F\pm}$ for some (in general non smooth) vector fields $Y^{F\pm}$ on \mathcal{K} . We have

$$\begin{aligned} (\text{dpr}_{\text{pr}(\kappa)}^t \circ \underline{\omega}_{\text{pr}(\kappa)}^\# \circ \text{dpr}_k) Y^{F\pm} &= (d\iota_{\iota(\kappa)}^t \circ \omega_{\iota(\kappa)}^\#) X_{\iota(\kappa)}^{F\pm} \\ &= d_{\mathcal{K}} f_{\pm} \\ &= \text{dpr}_{\text{pr}(\kappa)}^t d_{\mathcal{K}/G} f_{\pm} \quad , \end{aligned}$$

i.e. the Hamiltonian vector fields $\underline{X}^{f\pm}$ of $f_{\pm} \in \mathcal{C}^\infty(\mathcal{K}/G)$ are given by $\underline{X}_{\text{pr}(\kappa)}^{f\pm} = \text{dpr} Y_{\kappa}^{F\pm}$ and the Poisson bracket $\{ , \}_{\mathcal{K}/G}$ associated with $\underline{\omega}$ can be computed using extensions:

$$\begin{aligned} \text{pr}^* \{ f_+ , f_- \}_{\mathcal{K}/G} &:= \underline{\omega}(\underline{X}^{f_-} , \underline{X}^{f_+}) \circ \text{pr} \\ &= \text{pr}^* \underline{\omega}(Y^{F_-} , Y^{F_+}) \\ &= \omega(X^{F_-} , X^{F_+}) \circ \iota \\ &=: \iota^* \{ F_+ , F_- \}_{\Lambda} \quad . \end{aligned}$$

Let now $\{ , \}'$ be the Poisson bracket induced by the symplectic groupoid of (i) on P/G and $u_{\pm} \in \mathcal{C}^\infty(P)^G$; since $s_{\Lambda}^* u_{\pm} \in \mathcal{C}^\infty(\Lambda)$ extend $s_{\mathcal{K}}^* u_{\pm} \in \mathcal{C}^\infty(\mathcal{K})^G$,

$$\begin{aligned} \{ u_+ , u_- \}'([p]_{P/G}) &:= \{ s_{\mathcal{K}/G}^* u_+ , s_{\mathcal{K}/G}^* u_- \}_{\mathcal{K}/G}(\varepsilon_{\mathcal{K}/G}([p]_{P/G})) \\ &= \{ s_{\Lambda}^* u_+ , s_{\Lambda}^* u_- \}_{\Lambda}(\varepsilon_{\Lambda}(p)) \\ &= \{ u_+ , u_- \}_P(p) \\ &=: \{ u_+ , u_- \}_{P/G}([p]_{P/G}) \quad , \end{aligned}$$

for all $p \in P$, (ii) follows uniqueness (proposition 5.1). \square

Theorem 5.2 generalizes a result by Xu ([25], theorem 4.2), regarding Poisson actions with a complete momentum map. A momentum map [20] for a Poisson G -space (P, π) is a Poisson map $j : P \rightarrow G^*$ such that $\sigma(x) = \pi^\# j^* \overleftarrow{x}$, for the infinitesimal action $\sigma(\cdot) : \mathfrak{g} \rightarrow \mathfrak{X}(P)$ and the left invariant 1-form \overleftarrow{x} on G^* associated with $x \in \mathfrak{g} \simeq \mathfrak{g}^{**}$; such a Poisson map is called complete, when the Hamiltonian vector field of $j^* f$ is complete for all compactly supported $f \in \mathcal{C}^\infty(G^*)$. If G is 1-connected, so that the right dressing action of G on G^* is globally defined, j is always equivariant. The construction of [25], when G is complete and 1-connected, the action admits a complete momentum map j , produces a symplectic groupoid $J^{-1}(e_*)/G \rightrightarrows P/G$, assuming the quotient $J^{-1}(e_*)/G$ is smooth, where the momentum map $J : \Lambda \rightarrow G^*$, is given by $J(\lambda) = j(t(\lambda)) \cdot j(s(\lambda))^{-1}$. Note that J is by construction a morphism of Lie groupoids and one can check [26] that it differentiates to \hat{j} , therefore J coincides with our \mathcal{J} ; it is easy to see that the G action on $J^{-1}(e_*)$ of [25] is the same as that induced by the $\mathcal{S} \rightrightarrows G^*$ -action on Λ ($\mathcal{S} \simeq G^* \rtimes G$, under the assumptions). Specializing last theorem to the case considered in [25], shows that Xu's quotient is always smooth.

A Lie group G is trivially ($\Pi = 0$) a complete Poisson group, with the abelian group $\overline{\mathfrak{g}^*}$ as a dual Poisson group; in this case a Poisson group action is an action by Poisson diffeomorphisms and our approach of sections §3-§4 reproduces the “symplectization functor” treatment of Fernandes [7] and Fernandes-Ortega-Ratiu [8] from the viewpoint of double structures. The construction given in [8] (proposition 4.6) of a symplectic groupoid for the quotient Poisson manifold is precisely the construction of theorem 5.2 in the special case $\Pi = 0$. In the same paper, a condition ([8], theorem 4.11) for integration to commute with reduction is also given. This issue depends only on the connectivity of the source fibres of $\mathcal{J}^{-1}(e_*)/G \rightrightarrows P/G$ and the conditions of [8] apply to Poisson actions of Poisson groups. From the proof of theorem 5.2 it is clear that the Lie groupoid on $\mathcal{J}^{-1}(e_*)/G$ is really a “push-forward” of $\mathcal{J}^{-1}(e_*)$, obtained by identifying the source fibres along the G -orbits of P ; then $\mathcal{J}^{-1}(e_*)/G$ is source (1-)connected iff so is $\mathcal{J}^{-1}(e_*)$. For all $p \in P$, the quotient $\mathbb{K}_p(\hat{j})$ of the space of T^*P -loops taking values in $\ker \hat{j}$ which are T^*P -homotopic to the null T^*P -path 0_p modulo T^*P -homotopies taking values in $\ker \hat{j}$, is naturally a group.

Proposition 5.3. *Under the same hypotheses of theorem 5.2, the source connected component of $\mathcal{J}^{-1}(e_*)/G \rightrightarrows P/G$ is the source 1-connected symplectic groupoid of P/G iff the groups $\mathbb{K}_p(\hat{j})$ are trivial, for all $p \in P$.*

The proof of ([8], theorem 4.11) applies for $\Pi \neq 0$ with no modification.

Appendix A. End of proof of proposition 3.3

It remains to check the anchor compatibility condition. Recall that

$$\begin{aligned} X(\varepsilon, u) &= H(\varepsilon, u) \cdot H(0, u)^{-1} * \gamma_-(u) \\ &= H(\varepsilon, u) * H(0, u)^{-1} * \gamma_-(u) \quad , \quad u, \varepsilon \in I \quad , \end{aligned}$$

and set $a = \delta_r^u H$ and $b = \delta_r^\varepsilon H$. The derivative of X in the ε -direction is thus

$$\begin{aligned} \partial_\varepsilon X(\varepsilon, u) &= \Upsilon(\partial_\varepsilon H(\varepsilon, u))_{H(0, u)^{-1} * \gamma_-(u)} = \Upsilon(\mathrm{dr}_{H(\varepsilon, u)} b(\varepsilon, u))_{H(0, u)^{-1} * \gamma_-(u)} \\ &= \Upsilon(b(\varepsilon, u))_{X(\varepsilon, u)} \quad , \end{aligned}$$

since for all $g \in G$, $h^* \in G^*$, $\xi = \dot{g}^*(o) \in \mathfrak{g}^*$ and path g^* in G^*

$$\Upsilon(\mathrm{dr}_{h^*} \xi) = \left. \frac{d}{d\alpha} \right|_{\alpha=0} \tilde{\Upsilon}(g^*(\alpha) \cdot h^*, g) = \left. \frac{d}{d\alpha} \right|_{\alpha=0} \tilde{\Upsilon}(g^*(\alpha), h^* * g) = \Upsilon(\xi)_{h^* * g} \quad .$$

The computation of the derivative in the u -direction is more involved. We have

$$\partial_u X(\varepsilon, u) = \mathrm{d}\tilde{\Upsilon}_{(H(\varepsilon, u) \cdot H(0, u)^{-1}, \gamma_-(u))}(\delta H, \dot{\gamma}_-(u)) \quad ,$$

where $\dot{\gamma}_-(u) = \Upsilon(\xi_-(u))_{\gamma_-(u)} = \Upsilon((\iota_{\partial_H}^* - h)(u)) = \Upsilon(a(0, u))_{X(0, u)}$ and

$$\begin{aligned} \delta H &= d\mu_{(H(\varepsilon, u), H(0, u)^{-1})}^*(\partial_u H(\varepsilon, u), d\iota_* \partial_u H(0, u)) \\ &= d\mu_{(H(\varepsilon, u), H(0, u)^{-1})}^*(dr_{H(\varepsilon, u)} a(\varepsilon, u), d\iota_* dr_{H(0, u)} a(0, u)) \\ &= d\mu_{(H(\varepsilon, u), H(0, u)^{-1})}^*(0_{H(\varepsilon, u)}, d\iota_* dr_{H(0, u)} a(0, u)) \\ &+ d\mu_{(H(\varepsilon, u), H(0, u)^{-1})}^*(dr_{H(\varepsilon, u)} a(\varepsilon, u), 0_{H(0, u)^{-1}}) \\ &= \delta H_+ + \delta H_- \end{aligned}$$

since the tangent group multiplication $d\mu^* : TG^* \times TG^* \rightarrow TG^*$ is fibrewise linear, with

$$\begin{aligned} \delta H_+ &= dl_{H(\varepsilon, u)} d\iota_* dr_{H(0, u)} a(0, u) = dl_{H(\varepsilon, u)} dl_{H(0, u)^{-1}} d\iota_* a(0, u) \\ &= dl_{H(\varepsilon, u) \cdot H(0, u)^{-1}} d\iota_* a(0, u) \\ \delta H_- &= dr_{H(0, u)}^{-1} dr_{H(\varepsilon, u)} a(\varepsilon, u) = dr_{H(\varepsilon, u) \cdot H(0, u)^{-1}} a(\varepsilon, u) \end{aligned}$$

The tangent action map $d\tilde{\Upsilon} : TG^* \times TG \rightarrow TG$ is also fibrewise linear, hence

$$\begin{aligned} \partial_u X(\varepsilon, u) &= d\tilde{\Upsilon}_{(H(\varepsilon, u) \cdot H(0, u)^{-1}, \gamma_-(u))}(\delta H_+, \Upsilon(a(0, u))_{\gamma_-(u)}) \\ &+ d\tilde{\Upsilon}_{((H(\varepsilon, u) \cdot H(0, u)^{-1}, \gamma_-(u))}(\delta H_-, 0_{\gamma_-(u)}); \end{aligned}$$

the first term of last expression vanishes, since it can be rewritten as

$$\begin{aligned} &d\tilde{\Upsilon}_{@}(dl_{(H(\varepsilon, u) \cdot H(0, u)^{-1})} d\iota_* a(0, u), \Upsilon(a(0, u))_{\gamma_-(u)}) \\ &= d\tilde{\Upsilon}_{@}(dl_{H(\varepsilon, u) \cdot H(0, u)^{-1}} d\iota_* a(0, u), d\tilde{\Upsilon}_{(e_*, \gamma_-(u))}(a(0, u), 0_{\gamma_-(u)})) \\ &= d\tilde{\Upsilon}_{@}(d\mu_{(H(\varepsilon, u) \cdot H(0, u)^{-1}, e_*)}^*(dl_{(H(\varepsilon, u) \cdot H(0, u)^{-1})} d\iota_* a(0, u), a(0, u)), 0_{\gamma_-(u)}) \\ &= d\tilde{\Upsilon}_{@}(0_{H(\varepsilon, u) \cdot H(0, u)^{-1}}, 0_{\gamma_-(u)}) = 0 \end{aligned}$$

by equivariance, where we have set $@ = (H(\varepsilon, u) \cdot H(0, u)^{-1}, \gamma_-(u))$ to simplify the expressions; therefore

$$\begin{aligned} \partial_u X(\varepsilon, u) &= d\tilde{\Upsilon}_{((H(\varepsilon, u) \cdot H(0, u)^{-1}, \gamma_-(u))}(dr_{H(\varepsilon, u) \cdot H(0, u)^{-1}} a(\varepsilon, u), 0_{\gamma_-(u)}) \\ &= d\tilde{\Upsilon}_{(e_*, H(\varepsilon, u) \cdot H(0, u)^{-1} * \gamma_-(u))}(a(\varepsilon, u), 0_{H(\varepsilon, u) \cdot H(0, u)^{-1} * \gamma_-(u)}) \\ &= \Upsilon(a(\varepsilon, u))_{X(\varepsilon, u)} \end{aligned}$$

We just have shown that

$$\begin{aligned} dX_{(\varepsilon, u)} &= \Upsilon(\delta_r^\varepsilon H(\varepsilon, u))_{X(\varepsilon, u)} \cdot d\varepsilon + \Upsilon(\delta_r^u H(\varepsilon, u))_{X(\varepsilon, u)} \cdot du \\ &= \Upsilon(h(\varepsilon, u))_{X(\varepsilon, u)} \end{aligned}$$

and this concludes the proof.

Appendix B. Notations and conventions

A bullet, “•”, usually denotes the one point manifold and $\Gamma(f) = \{x, f(x)\}$ is the graph of a map f . For any vector bundle E and section $B \in \Gamma(\wedge^2 E)$ $B^\sharp : E^* \rightarrow E$ is the associated bundle map. The structural map of a typical groupoid are source s , target t , unit section ε , inversion ι and partial multiplication μ ; in special cases we use additional symbols, which should be clear from the context. We denote with $\mathcal{G}^{(n)} = \{(g_1, \dots, g_n) \mid s(g_i) = t(g_{i+1})\}$, $n \in \mathbb{N}$, the nerves of a groupoid or a differentiable graph $(\mathcal{G}, M; s, t)$, i.e. the strings of composable elements. As it is customary, we call a groupoid source 0- or 1-connected, if it has 0- or 1-connected source fibres, respectively. The conormal bundle of a submanifold $C \subset M$ is $N^*C \equiv \text{Ann } TC \subset T^*M$; if M is symplectic $T^\omega C$ denotes the symplectic orthogonal bundle. Differently from [22], in this paper we denote with \overline{P} the opposite Poisson manifold $(P, -\pi)$ of a given Poisson manifold (P, π) . The Hamiltonian vector field of $f \in \mathcal{C}^\infty(M)$ is $X^f = \{f, \cdot\}$ if M is symplectic or Poisson; in order to make this consistent we define the Poisson bracket on $\mathcal{C}^\infty(M)$ as $\{f, g\} = \omega(X^g, X^f)$, in the first case and $\{f, g\} = \pi(df, dg)$ in the second.

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